HW 6

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2(a) $3z^2+1=0 \iff z=\frac{i}{\sqrt{3}}$, which is inside the square

(b) $\sin(z/2)=0 \iff \frac{e^{i\frac{z}{2}}-e^{-i\frac{z}{2}}}{2i}=0 \iff \frac{e^{\frac{ix-y}{2}}-e^{\frac{-ix+y}{2}}}{2}=0$

$\iff e^{\frac{ix-y}{2}}=e^{\frac{-ix+y}{2}} \iff e^{ix-y}=e^0 \iff y=0$ and $\cos x=1$

$\iff y=0$ and $x=2n\pi,\ n\in\mathbb{Z}$,

which are inside the square or outside the circle.

(c) $1-e^z=0 \iff z=2n\pi i,\ n\in\mathbb{Z}$,

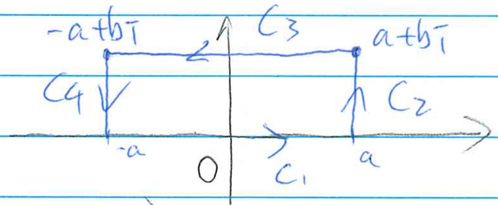
which are inside the square or outside the circle.

So in all of the above cases, the singularties lie outside of the region bounded by C_1 and C_2. In other words, f is analytic inside the region bounded by C_1 and C_2. By the corollary, the result follows.

3. Consider a large enough circle C_0 that encloses and does intersect the rectangle. Since the only possible singularity $2+i$ is inside the rectangle, $(z-2-i)^{n-1}$ is analytic inside the region bounded by C and C_0. So

$$\int_C (z-2-i)^{n-1}\,dz = \int_{C_0}(z-2-i)^{n-1}\,dz = \begin{cases} 0 & \text{when } n=\pm1,\pm2,\dots \\ 2\pi i & \text{when } n=0\end{cases}.$$

4(a) On C_1, $e^{-z^2}=e^{-x^2}$



On C_3, $e^{-z^2}=e^{-(x+bi)^2}$

$= e^{-x^2+b^2-2xbi}$

$= e^{b^2}e^{-x^2}(\cos 2xb - \sin 2xb)$

$$\int_{C_1} e^{-z^2}dz + \int_{C_3} e^{-z^2}dz = \int_{-a}^{a} e^{-x^2}dx - \int_{-a}^{a} e^{b^2}e^{-x^2}(\cos 2xb - \sin 2xb)\,dx$$

$$= 2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2xb,$$

Since e^{-x^2}, $e^{-x^2}\cos 2xb$ are even functions, and $e^{-x^2}\sin 2xb$ is an odd function.

(a) On C_2, $e^{-z^2} = e^{-(a+yi)^2} = e^{-a^2+y^2-2ayi}$

On C_4, $e^{-z^2} = e^{-(-a+yi)^2} = e^{-a^2+y^2+2ayi}$

$$\int_{C_2} e^{-z^2}dz + \int_{C_4} e^{-z^2}dz = \int_0^b e^{-a^2+y^2-2ayi}(i)\,dy + \int_0^b e^{-a^2+y^2+2ayi}(-i)\,dy$$

$$= ie^{-a^2}\int_0^b e^{y^2}e^{-2ayi}\,dy - ie^{-a^2}\int_0^b e^{y^2}e^{2ayi}\,dy$$

(b) Note that $\left|\int_{C_2}+\int_{C_4} e^{-z^2}dz\right| \le e^{-a^2}\left(\int_0^b |e^{y^2}e^{-2ayi}|\,dy + \int_0^b |e^{y^2}e^{2ayi}|\,dy\right)$

$\le 2e^{-a^2}\int_0^b e^{y^2}dy \to 0$ as $a \to +\infty$

Let $C = C_1+C_2+C_3+C_4$.

Since e^{-z^2} is an entire function,

$$\int_C e^{-z^2}dz = 0$$

So $\lim_{a\to\infty}\int_C e^{-z^2}dz = 0 \Rightarrow \overbrace{\lim_{a\to\infty}\left(\int_{C_2}+\int_{C_4}\right)e^{-z^2}dz}^{\to 0} + \lim_{a\to\infty}\left(\int_{C_1}+\int_{C_3}\right)e^{-z^2}dz = 0$

$\Rightarrow \lim_{a\to\infty} 2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx = 0$

$\Rightarrow \int_0^\infty e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^\infty e^{-x^2}dx$

$= \frac{\sqrt{\pi}}{2}e^{-b^2}$

5. By Cauchy-Goursat thm,

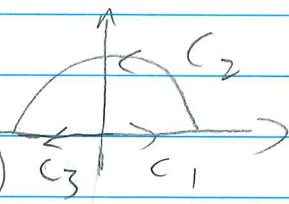
$\int_{C_1-C_3} f(z)dz = 0$ and $\int_{C_2+C_3} f(z)dz$, since f is entire and C_1-C_3 and C_2+C_3 are simply closed positively oriented contour.

So $\int_{C_1} f(z)\,dz = \int_{C_3} f(z)\,dz$ and $\int_{C_2} f(z)\,dz = -\int_{C_3} f(z)\,dz$

$\Rightarrow \int_{C_1} f(z)dz = -\int_{C_2} f(z)\,dz$

$\Rightarrow \int_{C_1+C_2} f(z)dz = 0 \Rightarrow \int_C f(z)\,dz = 0$.

6. $\int_{C_1} f(z)\,dz = \int_0^1 \sqrt{r}\,dr$ $\quad (z(r) = r)$



$= \frac{2}{3} r^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}$

$\int_{C_2} f(z)dz = \int_0^\pi e^{i\frac{\theta}{2}}(ie^{i\theta})\,d\theta$ $\quad (z(\theta) = e^{i\theta})$

$= i\int_0^\pi e^{i\frac{3\theta}{2}}\,d\theta$

$= \frac{2}{3}e^{i\frac{3\theta}{2}}\Big|_0^\pi$

$= \frac{2}{3}(-i-1)$

$\int_{C_3} f(z)dz = \int_0^1 \sqrt{r}\,e^{i\frac{\pi}{2}}(-1)\,d\theta$ $\quad (z(r) = -r)$

$= -\int_0^1 \sqrt{r}\,i\,d\theta$

6. $= -\frac{3}{2}i\,\sqrt{r}\,\Big|_0^1 = -\frac{3}{2}i$

So $\int_C f(z)\,dz = \left(\int_{C_1} + \int_{C_2} - \int_{C_3}\right) f(z)\,dz$

$= \frac{2}{3} + \frac{2}{3}(-i-1) - \left(-\frac{3}{2}i\right) = 0$.

Note that $f(z)$ is not differentiable at 0 (you may check this by definition), so it does not satisfy the requirement of Cauchy - Goursat theorem

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1. (a) $\int_C \frac{e^{-z}}{z-\pi i/2}\,dz = 2\pi i\, e^{-\frac{\pi}{2}i} = 2\pi i(-i) = 2\pi$

(b) $\int_C \frac{\cos z}{z(z^2+8)}\,dz = 2\pi i \,\frac{\cos 0}{0^2+8} = \frac{\pi i}{4}$

(c) $\int_C \frac{z\,dz}{2z+1}\,dz = \int_C \frac{1}{z+\frac{1}{2}}\,\frac{z}{2}\,dz = 2\pi i\left(\frac{-\frac{1}{2}}{2}\right) = -\frac{\pi i}{2}$

(d) $\int_C \frac{\cosh(z)}{z^4}\,dz = \frac{2\pi i}{3!}\,\frac{d^3}{dz^3}\cosh(z)\Big|_{z=0} = \frac{2\pi i}{3!}\sinh(0) = 0$.

(e) $\int_C \frac{\tan(\frac{z}{2})}{(z-x_0)^2}\,dz = \frac{2\pi i}{1!}\,\frac{d}{dz}\tan\left(\frac{z}{2}\right)\Big|_{z=x_0} = 2\pi i\cdot\frac{1}{2}\sec^2\left(\frac{z}{2}\right)\Big|_{z=x_0} = \pi i \sec^2\left(\frac{x_0}{2}\right)$

2. let C be the circle $|z-i| = 2$

(a) $\int_C g(z)\,dz = \int_C \frac{1}{z-2i}\cdot\frac{1}{z+2i}\,dz = 2\pi i\,\frac{1}{2i+2i} = \frac{\pi}{2}$

(b) $\int_C g(z)\,dz = \int_C \frac{1}{(z-2i)^2}\cdot\frac{1}{(z+2i)^2}\,dz$

$= 2\pi i\,\frac{d}{dz}\frac{1}{(z+2i)^2}\Big|_{z=2i}$

$= 2\pi i\left(-\frac{2}{(z+2i)^3}\Big|_{z=2i}\right)$

$= 2\pi i\left(-\frac{2}{(4i)^3}\right)$

$= \frac{\pi}{16}$

3. $g(2) = \int_C \frac{2s^2-s-2}{s-2}\,ds$

$= 2\pi i\,(2\cdot 2^2 - 2 - 2)$, since 2 is inside C

$= 8\pi i$

For $|z| > 3$, since $\frac{2s^2-s-2}{s-z}$ is analytic on and inside the circle, $g(z) = 0$ by Cauchy Goursat thm.

4. $g(z) = \int_c \frac{s^3+2s}{(s-z)^3}\, dz$

$= \frac{2\pi i}{2!} \frac{d^2}{ds^2}(s^3+2s)\Big|_{s=z}$

$= \pi i \, [6s]_{s=z}$

$= 6\pi i z$

$g(z) = 0$ when z is outside by similar reason is Q3.

10. By Cauchy inequality, $\forall z_0 \in \mathbb{C}$,

$$|f^{(2)}(z_0)| \leq \frac{2!}{R^2} \max_{z \in B_R(z_0)} |f(z)|$$

$$\leq \frac{2}{R^2} \max_{z \in B_R(z_0)} |z|$$

$$\leq \frac{2}{R^2} (|z_0| + R) \to 0 \text{ as } R \to \infty$$

So $f'(z)$ is a constant and $f(z) = a_1 z + a_0$.

Note that $|a_0| = |f(0)| \leq 0 \Rightarrow a_0 = 0$

So $f(z) = a_1 z$.